Math Circles - Intro to Combinatorics - Winter 2024

Solution Set 3

February 21th, 2024

1. We will start with looking at the diagonals of the triangle.
   (a) What do you notice about the first diagonal?
   Solution: The first diagonal is all ones.
   (b) What do you notice about the second diagonal?
   Solution: It's the natural numbers/counting numbers.
   (c) What do you notice about the third diagonal? Try to find a way to draw these numbers using dots in a pattern.
   Solution: These numbers have a special name. They are triangular numbers. We can use dots to draw equilateral triangles with exactly the number of dots of a triangle number. Try drawing some triangles to check.
   (d) Can you think of a way that the pattern in the third diagonal might extend to the fourth and fifth diagonal?
   Solution: It is a bit harder to imagine this, but for the fourth diagonal we have tetrahedral number, which are pyramids with the number of dots in three dimensions. The fifth diagonal is pentatope numbers, which are a similar concept in four dimensions. In fact the diagonals continue in this manner with the diagonals representing triangular based shapes in higher dimensions. In math we think of these numbers as representing an idea we call simplicies.
2. Do you notice anything if you mark all the even number?
   Solution: If you color in all the even numbers on the triangle you get a bunch of upside down triangles.
3. Or the odd numbers?
   Solution: The odd numbers give us a pattern of triangles inside of triangles. This pattern is an approximation of a famous fractal known as Sierpinski's triangle. One fun fact about this is that it shows that us that the proportion of odd numbers in the triangle approaches zero as the number of rows in the triangle approaches infinity.
4. What about number that are divisible by three? Or any other number?
   Solution: You get another pattern of triangles. In fact if you color any specific multiples you get some pattern of triangles. Maybe try some other multiples to see what sorts of triangle patterns you can find.
5. Do you notice anything special about the rows that correspond to prime numbers?
   Solution: Every entry, except for the one in that row is divisible by that prime number. This patter doesn't work for number that are composite(meaning not prime and having at least one factor other than itself and one).
6. What happens if you alternate adding and subtracting the entries in a row of pascals triangle?
   Solution: The sum is zero! For example, $1-4+6-4+1=8-8=0$.
7. Starting at any number in the pascals triangle, imagine walking through the hexagons. Starting at any number how many different paths can you find to get to the top most one?
   Solution: The number you start at tells you the number of paths you can walk! This is a direct consequence of each entery being the sum of two entries above it.

8. Imagine you shift the triangle so that it is left justified. This means writing the rows so the first entries all line up, then the second entries, and so on. Your picture should now look like a staircase, with the the top of the stairs on the left. Now take each diagonal and add up the entries. These numbers form a special sequence in math. Do you know it? Write out the sequence and try to find how each number is obtained from the previous numbers.
   Solution: The sequence we get is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$ The rule for the sequence is that we get the next number in our pattern by adding the two previous terms together. This is a very famous pattern called the Fibonacci sequence. Looking up this sequence for more fun patterns in numbers. It relates to the golden ratio and can be found in many natural phenomena. There is a video called "Nature by Numbers" on YouTube that shows some ways this pattern appears in nature.

9. Imagine each row of Pascal's triangle is a number. So we have 1, 11, 121, and so on. What pattern can you find in these numbers?
   Solution: Each row is 11^n. The first being 11^0. So if you want to find 11^n, you just need the 11th row, starting counting at zero of pascal's triangle.